

Petit's Polyhedron

David Richter

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Introduction. This ceramic model represents a realization of Boy's surface as an orthogonal polyhedron. This document is a guide for understanding this model.

Some history. In 1900, David Hilbert asked his student Werner Boy to prove that the real projective plane $\mathbb{R}P^2$ has no immersion in $\mathbb{R}^3$. As a matter of fact, Boy found an immersion which was thenceforth called Boy's surface. Fast-forward past two world wars, Stephen Smale proved the existence of an eversion of the sphere S^2 and subsequently a group including Arnold Shapiro, Anthony Phillips, and Bernard Morin constructed several eversions, with at least one using Boy's surface in a critical way. During this time, Morin became acquainted with an illustrator Jean-Pierre Petit, who assisted to provide visualizations for some of Morin's ideas. Amidst these efforts, Petite discovered this realization of Boy's surface as an orthogonal polyhedron, and subsequently illustrated it in his topological comic book, [7]. Thus, the first printed account of this polyhedron appeared in a comic book!

Polyhedral immersion. Petit's model is "polyhedral" in the sense that every facet is a flat polygon. It has six rectangles, six hexagons, and three octagons, with three polygons meeting at 90° dihedral angles at each of its 28 vertices. One must identify these polygons and see these incidences in order to understand the structure of this model.

Identifying some of the facets is complicated because several of them cut through the others. Notice that a square appears at the bottom of each of the apparent pits in the three petals of the model. Each such square is a section of one of the hexagonal facets of the polyhedron, formed by the intersection of this hexagon with two other hexagons, an octagon, and a rectangle. Having identified and accounted for all the faces, one should verify that the Euler characteristic is equal to 1.

That this model represents an "immersion" stems from the following observations. First, at every point, one may identify a small patch of the surface which contains the point and is homeomorphic to a disc. Second, at every place where the surface intersects itself, the disc-like patches containing the intersection point cross at 90° angles.

Triple point and propellor. According to a theorem of Tom Banchoff, every 3D immersion of $\mathbb{R}P^2$ has an odd number of triple points. This polyhedron has exactly one triple point, where three hexagonal facets intersect in mutually perpendicular planes. This coincides with the center of a "propellor" comprised of the three squares lying at the bottoms of the three pits, described above. In particular, the self-intersection locus of this immersion is a bouquet of three squares sharing the triple point as a vertex common to all three.

Regular map. The coloring of the facets shows a cellular decomposition of $\mathbb{R}P^2$ into six discs or "regions", where each disc shares a border with each of the others. The boundaries of

the regions are represented by uncolored narrow buffers. This allows one to see that $\mathbb{R}P^2$ has the sphere S^2 as a 2-to-1 covering space as follows. Recall that the regular dodecahedron represents a cellular decomposition of the sphere S^2 into twelve discs in which each disc shares a border with five other discs. This arrangement is invariant under the antipode map (mapping $v \mapsto -v$ for all v). If we quotient by this equivalence, then we obtain the regular map represented by these six regions. In other words, the preimage of each of these discs under this covering map is a pair of discs that are related to each other through the antipode map.

Petersen graph. If one can identify the six regions described above, then one should locate the 10 points where precisely three regions meet. Each region has five such points. These points and the 15 piecewise linear paths which connect them, i.e. the 1-dimensional skeleton of this cellular complex, comprise 3D orthogonal drawing of the Petersen graph, as embedded in the real projective plane. In fact, the Petersen graph cannot be embedded in $\mathbb{R}^3$ without forming a linked pair of cycles in the graph, known as a Hopf link. One should therefore seek such a pair of cycles.

Möbius tour. The surface of this polyhedron can be conceived as an assembly of 84 unit squares (which says a lot about how this model was assembled). In fact, there is a closed path on this surface which passes through each of the unit squares exactly once. It is an exercise to find such a path and check whether it yields a Möbius band embedded in the real projective plane.

Exercise. One cannot realize Boy's surface as an integral orthogonal polyhedron confined to the cube $[0, 3]^3$.

References

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