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On the Early History of Bessel Functions

JACQUES DUTKA

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1. Introduction

Linear differential equations of the second order are of fundamental importance in physical and technological applications. Among the most frequently encountered of such equations is Bessel's equation, defined by

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \quad (1)$$

where ν is an arbitrary complex parameter. The solutions of this equation, termed Bessel functions, are perhaps the most extensively tabulated of the higher transcendental functions. As technology and applied physics continue to develop, new applications of Bessel functions grow apace.

But Bessel functions are also important in pure mathematics in connection with problems in number theory, integral transforms, the evaluation of integrals, the theory of differential equations, *etc.* They were first systematically investigated by F. W. BESSEL [2], perhaps the outstanding astronomer of the nineteenth century, in 1824. However, for about a century previously, particular cases were obtained in important researches in mechanics, astronomy, and the conduction of heat by D. BERNOULLI, L. EULER, J. L. L. LAGRANGE, J. B. J. FOURIER, S. D. POISSON and others.

The most extensive treatise on Bessel functions is that of G. N. WATSON [1] which will be taken as the standard reference here. This monumental work gives a detailed and well-nigh exhaustive exposition of the subject as developed up to the early part of this century with numerical tables and an extensive bibliography. WATSON [1; Ch. I] also gives a historical sketch of the subject, including the work of Bessel, but concentrates primarily on the mathematical results without a detailed analysis of the interesting physical problems which led to these results. A brief discussion of the history of Bessel functions was sketched by G. A. MAGGI [1], and a more extensive treatment, dealing essentially with developments in the first half of the nineteenth century, was given by C. WAGNER [1]. The prodigious report of H. BURKHARDT [1] contains valuable summaries of the problems considered, and the results obtained by early investigators of Bessel functions.

The principal purpose of this paper is to supplement these accounts by giving a more detailed and connected account of the physical problems which led to these developments. A secondary purpose is to call attention to the

important but little known work of Bessel on trigonometric series, *published independently of and prior to the treatise of J. B. J. Fourier.*

In many cases, the original notation has been modernized and follows that of WATSON [1]. The original proofs of the results derived by early investigators have often been replaced by simpler or briefer treatments to make them more accessible to modern readers.

2. Some properties of Bessel functions

In the literature, Bessel functions are most frequently obtained by applying FROBENIUS' method for the solution of linear differential equations in the form of series, to (1.1). But other approaches are also utilized including recursive functional equations, integral representations, confluent hypergeometric functions, limits of solutions of second order linear differential equations, *etc.* They have also been obtained as particular cases from unified theories of special functions by C. TRUESDELL [1] and W. MILLER [1]. Each of these approaches has advantages in special applications.

Here Bessel functions will be derived by the method of generating functions stemming from P. A. HANSEN (1843) and O. SCHLÖMILCH [1] in 1857.

The series $\exp(xt/2) = \sum_{k=0}^{\infty} (xt/2)^k/k!$ and $\exp(-x/2t) = \sum_{k=0}^{\infty} (-1)^k(x/2t)^k/k!$ converge absolutely for all values of t , except for the second series at $t = 0$. Their product converges absolutely for all non-zero values of t . Thus one gets the generating function development

$$\exp \frac{x}{2} (t - t^{-1}) = \sum_{n=-\infty}^{+\infty} t^n J_n(x) \quad (1)$$

in which the term involving t^k in the first series is associated with the term involving t^{n-k} in the second series and the series on the right converges absolutely for all non-zero values of t . Thus for the coefficients of t^n and t^{-n} , one has

$$J_n(x) = (x/2)^n \sum_{k=0}^{\infty} (-1)^k \frac{(x/2)^n}{k!(n+k)!}, \quad (J_0(0) = 1, J_n(0) = 0; n \neq 0) \quad (2)$$

and

$$J_{-n}(x) = (-1)^n J_n(-x). \quad (3)$$

These are called Bessel functions of the first kind and integral order. On differentiating (1) with respect to x and t respectively, one gets on equating powers of t

$$\begin{aligned} J_{n-1}(x) - J_{n+1}(x) &= 2J'_n(x), \\ J_{n-1}(x) - J_{n+1}(x) &= (2n/x)J_n(x). \end{aligned} \quad (4)$$

It follows from the second of these relations that the functions of all integer orders can be calculated when two adjacent functions, say $J_0(x)$ and $J_1(x)$, are known. On adding and subtracting these equations, one gets

$$\begin{aligned} xJ'_n(x) + nJ_n(x) &= xJ_{n-1}(x), \\ xJ'_n(x) - nJ_n(x) &= -xJ_{n+1}(x). \end{aligned} \tag{5}$$

On multiplying by x^n and x^{-n} , respectively, one obtains

$$\begin{aligned} \frac{d}{dx} [x^n J_n(x)] &= x^n J_{n-1}(x), \\ \frac{d}{dx} [x^{-n} J_n(x)] &= -x^{-n} J_{n+1}(x). \end{aligned} \tag{6}$$

Rewrite these equations in the form

$$\begin{aligned} \frac{d}{dx} [x^n J_n(x)] &= x^n J_{n-1}(x), \\ \frac{d}{dx} [x^{1-n} J_{n-1}(x)] &= -x^{1-n} J_n(x). \end{aligned} \tag{6'}$$

and eliminate $J_{n-1}(x)$ to obtain Bessel's differential equation for $J_n(x)$:

$$x^2 \frac{d^2 J_n(x)}{dx^2} + x \frac{dJ_n(x)}{dx} + (x^2 - n^2) J_n(x) = 0. \tag{7}$$

When n is replaced by an arbitrary complex number ν , one gets the homogeneous second order differential equation

$$x^2 y'' + xy' + (x^2 - \nu^2)y = 0 \tag{8}$$

named for Bessel. (Cf. (1.1).) In the older literature, the solutions of this equation are sometimes called "Fourier-Bessel functions" or "cylindrical functions", the latter name arising from their frequent occurrence in the analysis of physical problems involving circular cylinders.

The general solution of (8) is

$$y = AJ_\nu(x) + BY_\nu(x), \tag{9}$$

where

$$\begin{aligned} J_\nu(x) &= (x/2)^\nu \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m!(\nu+m)!}, \\ Y_\nu(x) &= \frac{1}{\sin \nu\pi} [J_\nu(x) \cdot \cos \nu\pi - J_{-\nu}(x)] \end{aligned} \tag{10}$$

and A and B are arbitrary constants.

3. On James Bernoulli's problem

As early as 1694, JOHN BERNOULLI (1667–1748) came upon the nonlinear first order differential equation

$$y' = x^2 + y^2 \quad (1)$$

in connection with work on curves, but was unable to solve it. (This is a particular example of a class of equations investigated in the early eighteenth century by Count JACOPO RICCATI (1676–1754) and which is named for him.) JAMES BERNOULLI (1654–1705), the older brother (and rival) of JOHN finally succeeded in solving this equation, after many fruitless efforts. The solution was in the form of an infinite power series in x , and was given in a letter to G. W. LEIBNIZ [1; Vol. III, 74–75] of 3 October, 1702. This solution appears to be the earliest example in the mathematical literature of a result reducible to Bessel functions.

In effect, JAMES introduced a new variable u in (1) defined by

$$y = \frac{-1}{u} \frac{du}{dx} \quad (2)$$

and thus obtained a linear second order homogeneous equation

$$u'' + x^2u = 0 \quad (3)$$

which he proceeded to solve by the method of series. Assume $u = \sum_{n=0}^{\infty} a_n x^n$. Then on substituting in (3), one gets

$$\begin{aligned} u &= a_0 \left(1 - \frac{x^4}{4 \cdot 1! \cdot 3} + \frac{x^8}{4^2 \cdot 2! \cdot 3 \cdot 7} - \frac{x^{12}}{4^3 \cdot 3! \cdot 3 \cdot 7 \cdot 11} + \cdots \right) \\ &\quad + a_1 \left(x - \frac{x^5}{4 \cdot 1! \cdot 5} + \frac{x^9}{4^2 \cdot 2! \cdot 5 \cdot 9} - \frac{x^{13}}{4^3 \cdot 3! \cdot 5 \cdot 9 \cdot 13} + \cdots \right) \\ &= a_0 S_0(x) + a_1 S_1(x) \end{aligned} \quad (4)$$

in which a_0 and a_1 are arbitrary constants. If this is substituted in (2) with $a_0 \neq 0$, $a_1 = 0$, then one gets James Bernoulli's solution by division

$$y = -\frac{S'_0(x)}{S_0(x)} = \frac{x^3}{3} + \frac{x^7}{3^2 \cdot 7} + \frac{2x^{11}}{3^3 \cdot 7 \cdot 11} + \frac{13x^{15}}{3^4 \cdot 5 \cdot 7 \cdot 11} + \cdots \quad (5)$$

(Cf. WATSON [1; 2].)

The form of the coefficients of the powers of x in (4), involving products of factorials in the denominators, suggests a relation with Bessel functions (2.10). Indeed, E. C. J. VON LOMMEL (1837–1899) obtained a generalized form of BESSEL'S differential equation which is satisfied by $y = x^\alpha J_\nu(\beta x^\gamma)$ where α , β , γ are constants:

$$x^2 y'' - (2\alpha - 1)xy' + [(\beta\gamma x^\gamma)^2 + (\alpha^2 - \nu^2\gamma^2)]y = 0 \quad (6)$$

(WATSON [1; 97]). The solution may be written in the form

$$y = x^\alpha [AJ_\nu(\beta\gamma x^\gamma) + BJ_{-\nu}(\beta\gamma x^\gamma)] \quad (7)$$

where A and B are arbitrary constants. From this it is seen that

$$u = \sqrt{x} [AJ_{1/4}(x^2/2) + BJ_{-1/4}(x^2/2)] \quad (8)$$

is a solution of (3). (There is a typographical error in WHITTAKER & WATSON [1; 356] where it is stated that the Bessel function is of order $1/3$, and this has been repeated by some authors.)

From (2) and (8) it follows that

$$y = -\frac{u'}{u} = \frac{x[CJ_{-\frac{1}{4}}(z) + J_{\frac{1}{4}}(z)]}{J_{-\frac{1}{4}}(z) - CJ_{\frac{1}{4}}(z)}, \quad z = \frac{x^2}{2}, \quad (9)$$

where $C = -A/B$. On setting $C = 0$, one obtains (5).

Following the discovery of the calculus, mathematicians in the latter part of the seventeenth century devoted much attention to the solution of differential equations and by the next century virtually all the classical methods for solving first order differential equations had been developed. In the early eighteenth century, equations of higher order began to be studied.

(Count RICCATI investigated curves whose radii of curvature are functions only of their ordinates and was thus led to a non-linear first order equation of the form

$$y' + ay^2 = bx^\alpha \quad (10)$$

where a and b are constants and α is not necessarily an integer. The equation attracted the attention of JOHN BERNOULLI's family. His second son, DANIEL (1700–1782) showed in 1724 that the equation is integrable in finite terms when α is of the form $-4k/(2k \pm 1)$ where k is a non-negative integer.)

4. Some physical investigations in the Post-Renaissance era

In Europe, there was an efflorescence in the development of music in the seventeenth century. This was complemented by empirical and experimental studies of the physical as well as the aesthetic properties of musical instruments and their interdependence. In parallel with this, many scientists, building on studies made in antiquity as well as later developments in the medieval period, made notable advances. In particular, the monochord, already considered by the school of PYTHAGORAS in the sixth century B.C. (in the development of the laws of musical harmony) and their successors, received special attention, and notable advances were made. The dependence of the frequency of vibration of a stretched string on its length, tension and density was clarified and phenomena such as the presence of overtones and nodes, *etc.*, in a vibrating string were discovered or rediscovered. M. MERSENNE (1588–1648) found that the tensions and densities of strings being assumed equal, their fundamental

frequencies of vibration were inversely proportional to their lengths and directly proportional to the square root of their thicknesses.

One of the earliest mathematical contributions to the newly developed science of acoustics (the term was coined by JOSEPH SAUVEUR), was the memoir of BROOK TAYLOR (1685–1731) in 1713 [1], who gave a dynamical solution of the problem of the vibrating string. He supposed that a continuous stretched string, of length l and total mass ml , fastened at both ends, was composed of an “infinite” number of equal particles. In effect, by a consideration of the forces acting on a differential element of the string, he obtains an equation of the form $a^2\ddot{x} = \dot{s}y\dot{y}$ where \dot{s} is the time derivative of the arc length. At any time the string has the form $y = A \sin(x/a)$ and $y = A$ when $x = l/2$ so that $a = l/\pi$. Then, for the fundamental frequency of oscillation, TAYLOR finds $v = (1/2l)\sqrt{T/\sigma}$ where T denotes the tension and $\sigma = m/g$.

In 1727, JOHN BERNOULLI sent a letter to his son DANIEL in which he included the analysis of the motion of a vibrating string in the case in which n equidistantly spaced particles, each having the same mass, are suspended from a string of negligible mass. (This model had previously been considered by CHRISTIAAN HUYGENS (1629–1695).) If the spacing of the particles is sufficiently small, then it approximates a continuously loaded string. HUYGENS notes the isochronal nature of the oscillations in which each point of the string passes through its equilibrium position at the same time. JOHN BERNOULLI derived the fundamental frequency of the system whose equations of motion are of the form

$$\frac{d^2}{dt^2} - y_k = \left(\frac{n^2 T}{lM}\right) \cdot (y_{k+1} - 2y_k + y_{k-1}), \quad k = 1, 2, \dots, (n-1), \quad (1)$$

where T and M denote the tension of the string and the total mass of the particles respectively, for the cases $n = 1, 2, \dots, 6$. (Neither TAYLOR nor JOHN BERNOULLI considered frequencies of other types of oscillations which was actually done by DANIEL BERNOULLI. (See BURKHARDT [1; 1–3, 9] and C. TRUESDELL [2; 129ff.].))

The solutions of (1) can be expressed in terms of time in the form $y_k(t) = u_k \cdot f(t)$, where on substituting in (1), one finds that

$$\frac{d^2 f(t)}{dt^2} / f(t) = \left(\frac{n^2 T}{lM}\right) (u_{k+1} - 2u_k + u_{k-1}) / u_k$$

so that both members equal to a constant. A further detailed analysis is beyond the scope of this paper, but R. E. LANGER [1; Ch. 3] gives the solution

$$y_k(t) = a_v \sin \frac{kv\pi}{n} \cdot \cos \left(2t \sqrt{\frac{n^2 T}{lM}} \cdot \sin \frac{v\pi}{2n} \right); \quad k \text{ and } v \text{ integers.} \quad (2)$$

For further details see TRUESDELL [2; 132 ff] and CANNON & DOSTROVSKY [1; Ch. 8].

The analysis of a variant of the vibrating string problem by JOHN BERNOULLI's son, DANIEL, led to the first appearance of a particular Bessel function in a physical context.

5. Daniel Bernoulli's problem of the oscillations of a hanging chain

The earliest dynamical problem in which what are now termed "Bessel functions" appeared, was in an investigation by DANIEL BERNOULLI (1700–1782) in two memoirs [2] and [3] in 1732. He considered a heavy flexible uniform chain, suspended from a fixed point and free at its lower end. When the lower end is slightly disturbed, the chain is assumed to oscillate in a vertical plane about its position of equilibrium. DANIEL BERNOULLI sought the form of the curve of the chain at any time.

His approach was to regard the chain as composed of n linked pendulums in which the links and the weights suspended from the lower ends of the links are equal. He extended the methods of BROOK TAYLOR and JOHN BERNOULLI for the vibrating string by setting up a condition expressing the equilibrium of the forces involved and obtained the continuous case as a limit of the discrete model by letting n become infinite. A different approach, (stemming from EULER (1707–1783) in two memoirs [7] and [8]) more in accord with the modern viewpoint, will be given here.

Let l and ρ denote the length and linear density of the chain respectively. Let OX denote the vertical axis (upward) and OY the horizontal axis where O is the position of stable equilibrium of the lower end. Consider a point P on the chain with abscissa x . The tension T at P is ρgx , the weight of the chain below P . A differential element PQ of the chain is of length δs and mass $\rho \delta s$. For a slight disturbance of the chain, $dy/dx \ll 1$, so that $\delta s \simeq \delta x$. To a first order approximation, the vertical component of the resultant of the tensions at P and Q is zero, and its horizontal component is

$$\frac{\partial}{\partial x} \left(T \frac{\partial y}{\partial x} \right) \delta x.$$

The equation for the horizontal motion is

$$\rho \delta x \frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x} \left(\rho gx \frac{\partial y}{\partial x} \right) \delta x,$$

or

$$\frac{\partial^2 y}{\partial t^2} = g \frac{\partial}{\partial x} \left(x \frac{\partial y}{\partial x} \right). \quad (1)$$

Assume the solution of this equation is of form $y = u(x) \cdot T(t)$, where u and T are twice differentiable functions of x and t respectively. On substituting in (1) one has

$$\frac{\ddot{T}}{T} = \frac{g}{u} \frac{\partial}{\partial x} \left(x \frac{du}{dx} \right).$$

If the constant $-\omega^2$ denotes the common value of both members (where the negative sign corresponds to the fact that the motion is periodic and bounded),

one gets the equations

$$\ddot{T} = -\omega^2 T, \quad x \frac{d^2 u}{dx^2} + \frac{du}{dx} + \frac{\omega^2}{g} u = 0. \quad (2)$$

The solution of the first equation is $T = \cos(\omega t + \varphi)$ and the second was solved by BERNOULLI by the method of series (BERNOULLI [2; 116] and [3; 172]);

$$u = a_0 \left(1 - c^2 x + \frac{c^4 x^2}{2^2} - \frac{c^6 x^3}{2^2 \cdot 3^2} + \frac{c^8 x^4}{2^2 \cdot 3^2 \cdot 4^2} - \dots \right) \quad (3)$$

where $c^2 = \omega^2/g$. (c^2 is the reciprocal of the length of the simple isochronous pendulum corresponding to the system.) Now the differential equation for u is a particular case of LOMMEL's equation (3.6) whose general solution is of the form

$$u = AJ_0(2c\sqrt{x}) + BY_0(2c\sqrt{x}) \quad (4)$$

where A and B are constants. (A slightly more general form of the differential equation for u was later solved by L. EULER [3; ¶ 977].) At the free end of the chain, u must be small since the oscillations are small, so that $B = 0$. Thus one gets as the solution for y

$$y = CJ_0(2c\sqrt{x}) \cdot \cos(\omega t + \varphi), \quad J_0(2c\sqrt{l}) = 0 \quad (5)$$

Any root of the second equation determines a particular value of c and a proper frequency of oscillation about OX, the axis of stationary equilibrium. The constants C and φ are determined from the position and velocity of the free end at $t = 0$.

BERNOULLI [2; 116] assumed that $J_0(2c\sqrt{l})$ has an infinite number of real zeros. To solve for some of the smaller values, he employed an extension of a method which he had developed in 1728 for determining the largest (smallest) root of a finite algebraic equation (BERNOULLI [1]). (The correctness of BERNOULLI's assumption was proved by BESSEL [3] in 1824. Actually if $\nu > -1$, $J_\nu(x)$ can only have real zeros. (WATSON [1; 483]).) The zeros of $J_0(2c\sqrt{l})$ yield the angular frequencies of the system, and from this the proper vibrational modes. If ω_m denotes the m^{th} angular frequency and α_m , the m^{th} zero of $J_0(x)$, $m = 1, 2, 3, \dots$, then $\alpha_m = 2\omega_m\sqrt{l/g}$ and since (1) is a linear equation, it follows that the general solution of the problem is obtained by superposition:

$$y = \sum_{m=1}^{\infty} J_0\left(\alpha_m \sqrt{\frac{x}{l}}\right) \left(A_m \cos \sqrt{\frac{g}{l}} \frac{\alpha_m t}{2} + B_m \sin \sqrt{\frac{g}{l}} \frac{\alpha_m t}{2} \right). \quad (6)$$

DANIEL BERNOULLI made experiments with a chain consisting of short links loaded with small masses and partially confirmed some of the important physical implications of his results for the theory of vibrating systems. These marked a considerable advance over the results of his predecessors.

He also considered other types of boundary conditions in which a uniform chain of length l is suspended from a string of length λ of negligible mass, *etc.*, as well as a special case of a non-uniform chain.

DANIEL BERNOULLI's work was shortly developed further by L. EULER [1] who employed similar analytical methods. For the case of n equal weights uniformly spaced on a vertically suspended cord of length l , if $y_k = y_k(x)$ denotes the transverse displacement of the k^{th} weight from the lower end of the cord, $0 \leq k \leq n - 1$, EULER obtains an equilibrium condition equivalent to

$$\frac{1}{c^2} \left[\left(km \frac{l}{n} \right) \frac{y_{k+1} - 2y_k + y_{k-1}}{(l/n)^2} + m \frac{y_{k+1} - y_k}{(l/n)} \right] = -my_k, \quad y_n = 0, \quad (7)$$

where $m = \rho l/n$ for the linked pendulum. This, in effect, is the recursion relation of what were later termed the "Laguerre polynomials"

$$(k + 1)L_{k+1}(x) - (2k + 1 - x)L_k(x) + kL_{k-1}(x) = 0 \quad (8)$$

with

$$L_k(x) = \sum_{j=0}^k \binom{k}{k-j} \frac{(-x)^j}{j!}. \quad (9)$$

It was solved by EULER in the form

$$y_k = y_0 \cdot L_k(l/na), \quad (10)$$

where a is such that $L_n(l/na) = 0$. (As $n \rightarrow \infty$, one finds that $L_k(x/l) \rightarrow J_0(2\sqrt{x})$. The relation between the zeros of the Laguerre polynomials and the Bessel functions is discussed by G. SZEGÖ [1; 123f].) EULER confirms BERNOULLI's results in the continuous case, and also considers a non-uniform chain in which the density from the lower end varies as x^n . He obtains the equivalent of

$$\frac{x}{n+1} \frac{d^2u}{dx^2} + \frac{du}{dx} + c^2u = 0. \quad (11)$$

EULER [1; ¶ 21ff] points out that for $n = -1/2$, this is a RICCATI equation. He sets $q = -(n+1)c^2x$ and obtains a result equivalent to

$$u = Aq^{-n/2} I_n(2\sqrt{q}), \quad (12)$$

where $I_n(2\sqrt{q})$ is a modified Bessel function of order n . (These functions may be obtained from the generating function development $\exp x(t + t^{-1})/2 = \sum_{n=-\infty}^{+\infty} t^n I_n(x)$. Cf. (2.1).)

The memoirs of BERNOULLI [2] and [3] have been reproduced, with English translations, by CANNON & DOSTROVSKY [1; Appendix]. A detailed discussion of the physical interpretations of the foregoing results is given by TRUESDELL [2; ¶ 23].

About four decades later, EULER [7] and [8] returned to the problem of the oscillations of a hanging chain. He sets up the equations of motion for a chain of uniform density, ($n = 0$ in (12) above). To eliminate the constant c , let us put $z = c^2x$; we get

$$\frac{d}{dz} \left(z \frac{du}{dz} \right) + u = 0. \quad (2')$$

By applying the method of series, EULER obtained a solution, $u = J_0(2\sqrt{z})$ which satisfies the boundary condition $(du/dz + u) = 0$ at the free end where $z = 0$. (The functions $C_n = x^{\frac{n}{2}} J_n(2\sqrt{x})$ often arise in physical problems where they lead to convenient simplifications. They have been called "Clifford functions" after the English mathematician W. K. CLIFFORD (1845–1879). See his *Mathematical Papers* *XXXVII.) The general solution of (4) is $u = AJ_0(2\sqrt{z}) + BY_0(2\sqrt{z})$. EULER [7; ¶ 22] gives the equivalent of

$$u = J_0(2\sqrt{z})(A + B) \int \frac{dz}{z \cdot J_0^2(2\sqrt{z})} \quad (4')$$

(Cf. EULER [3; ¶ 977] where $Y_0(2\sqrt{z})$ is given in series form.) At the upper end of the chain $u = 0$ whence $B = 0$ and the problem reduces to the solution of $J_0(2c\sqrt{l}) = 0$ as in (5).

EULER then gives a very ingenious method of approximating the smallest zeros of $J_0(2\sqrt{z})$, $r_1 < r_2 < r_3 < \dots$ where $r_1 > 0$. (Cf. BERNOULLI'S method above.) He considers

$$J_0(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-z)^n}{(n!)^2} = \prod_{n=1}^{\infty} \left(1 - \frac{z}{r_n}\right)$$

where the product on the right is assumed to converge. Then one gets

$$\frac{d}{dz} \log J_0(2\sqrt{z}) = \sum_{n=1}^{\infty} \frac{1}{r_n - z} = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{r_n} \left(\frac{z}{r_n}\right)^m.$$

For $|z| < r_1$, the last series converges absolutely. Thus

$$\frac{d}{dz} J_0(2\sqrt{z}) = J_0(z) \sum_{m=0}^{\infty} s_{m+1} z^m, \quad \text{where } s_{m+1} = \sum_{n=1}^{\infty} r_n^{-(m+1)}$$

On equating like powers of z on both sides of the equation, one gets

$$s_1 = 1, \quad s_2 = 1/2, \quad s_3 = 1/3, \quad s_4 = 11/48, \quad s_5 = 19/120, \dots$$

Since $r_1^{-m} < s_m$ and $s_{m+1} < s_m/r_1$, one has

$$s_m^{-1/m} < r_1 < s_m/s_{m+1}, \quad m = 1, 2, 3, \dots$$

After a few steps EULER gets $r_1 \simeq 1.445795$ to six decimals (¶ 27). By separating out the powers of the approximation of r_1 , he gets $r_2 \simeq 7.6658$ and then $r_3 \simeq 18.63$. (Cf. WATSON [1; 500–507] and TRUESDELL [1; 317–319].)

EULER [8] discusses the oscillations of a string under its own weight. He considers the motion of a heavy taut horizontal cord with fixed ends and gets a solution of the form $y = AJ_0(2\sqrt{u}) + BY_0(2\sqrt{u})$ in which a series for $Y_0(2\sqrt{u})$ is developed. (For additional details, see TRUESDELL [2; 319–320] or M. KLINE [1; 502–508].)

6. Vibrating strings and membranes

About the middle of the eighteenth century, a famous controversy arose concerning the analytical form of the solution of the partial differential equation for the vibrations of a taut horizontal string. From (4.1), on replacing y_k by $y(t, x)$ and l/n by Δx , but keeping the tension constant during the vibrations, J. LE R. D'ALEMBERT (1717–1783) obtained an equation of the form

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \left[\frac{y(t, x + \Delta x) - 2y(t, x) + y(t, x - \Delta x)}{(\Delta x)^2} \right], \quad a^2 = \text{const.}$$

As $n \rightarrow \infty$ and $\Delta x \rightarrow 0$, he obtained

$$\frac{\partial^2 y(t, x)}{\partial t^2} = a^2 \frac{\partial^2 y(t, x)}{\partial x^2} \quad (1)$$

for the motion of the string. The boundary and initial conditions are

$$y(t, 0) = y(t, l) = 0; \quad \frac{\partial y}{\partial t}(0, x) = 0 \quad \text{and} \quad y(0, x) = f(x) \quad (2)$$

so that the horizontal string, fixed at both ends, is initially at rest with a form $y = f(x)$. The solution to this boundary value problem was found by D'ALEMBERT to be in the form

$$y(t, x) = \frac{1}{2} [\varphi(at + x) - \varphi(at - x)] \quad (3)$$

where φ is an arbitrary odd periodic function of time with period $2l$, twice differentiable, and $\varphi(x) = f(x)$, $0 \leq x \leq l$. Shortly thereafter, EULER, while accepting D'ALEMBERT's solution in general, differed with him on what types of initial configuration $f(x)$ are possible. EULER allows as a particular case a function of the form $f(x)$ which is not twice differentiable, indeed may be "irregular" (in the modern sense, continuous with discontinuous derivatives). That was not accepted by D'ALEMBERT. In 1753, DANIEL BERNOULLI, basing his argument on physical considerations, *viz.*, that a single vibration had the form $y = A \sin n\pi x/l \cdot \cos n\pi at/l$ (a form already given by B. TAYLOR in a tract of 1715 on finite differences), and that small oscillations could coexist, claimed that the general solution of the problem was

$$y(t, x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l} \quad (4)$$

which is a formal solution of (1) and (2) when $f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$. (As early as 1733, DANIEL BERNOULLI had published a tract on music in which he discussed overtones, and claimed that all oscillations are of a sinusoidal character.) But BERNOULLI's solution was not accepted by D'ALEMBERT or EULER, on differing grounds. In retrospect, it is curious that *formally* D'ALEMBERT's solution, with its mathematical basis (3), follows from BERNOULLI's solution (4), obtained on

physical grounds. For from the trigonometrical identity

$$\sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi at}{l} = \frac{1}{2} \left[\sin \frac{n\pi}{l} (at + x) - \sin \frac{n\pi}{l} (at - x) \right];$$

on substituting in (4), one gets

$$\begin{aligned} y(t, x) &= \frac{1}{2} \sum_{n=1}^{\infty} b_n \left[\sin \frac{n\pi}{l} (at + x) - \sin \frac{n\pi}{l} (at - x) \right] \\ &= \frac{1}{2} [\varphi(at + x) - \varphi(at - x)]. \end{aligned}$$

In 1759, J. L. L. LAGRANGE (1736–1813) entered the controversy (BURKHARDT [1; 29–31]), which had already lasted for about a decade, accepting EULER's solution as the most general, but differing with EULER on the latter's method of proof. He proposed to obtain a solution by first considering the case of a finite number of particles loading a weightless string, and then letting the number of particles become infinite. In the course of his investigations, he came upon expressions which are very close to those later obtained by FOURIER, but did not in fact make the change of order of summation and integration which would have been required.

The controversy, which basically involved the different meanings of the term "function" for the participants, and the lack of an adequate theory of convergence of infinite series, remained unresolved for many years.

For further details of the controversy see BURKHARDT [1; 10–47], TRUESDELL [1; Part III], K. KLINE [1; 502–508], I. GRATTAN-GUINNESS [1; Ch. 1].

After the detailed investigation of the vibrating string, it was natural to extend the problem to two dimensions – the transverse vibrations of a stretched membrane with uniform tension maintained along its boundary during the motion of the membrane. The membrane is regarded as a thin flexible lamina of uniform density throughout, and the tension per unit length of the boundary acts perpendicularly to the boundary.

EULER [4] made a difficult and important advance by considering the vibrations of stretched membranes. He regarded the membrane as being like a piece of cloth-composed of "infinite" numbers of uniformly spaced threads along its length (X) and width (Y). The displacement z of a point (x, y) is a function of x, y and t . EULER sets up the equations of motion for a homogeneous membrane in which the tensions in the X and Y directions are equal with no external forces acting on it. (The physical analysis of the problem is treated in texts on boundary value problems or acoustics and will not be discussed further here.) He obtains an equation of the form

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) \quad (5)$$

where the right member of the equation is proportional to the sum of the forces in the X and Y directions, and a^2 depends on the tension and density of the membrane.

EULER's solution for the case of a rectangular membrane is proportional to a product of sinusoidal functions of x , y and t respectively. He then proceeds to consider the case of a circular membrane in which the tension along the boundary is uniform. He introduces what are essentially cylindrical coordinates (r, θ, z) in place of the rectangular coordinates and substitutes in (5). After a straightforward but lengthy calculation, EULER [4, ¶ 15] finds

$$\frac{\partial^2 z}{\partial t^2} = a^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r} \frac{\partial z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 z}{\partial \theta^2} \right). \quad (6)$$

Assume the solution is of the form $z(r, \theta, t) = u(r)v(\theta)T(t)$. Then one obtains the equations

$$\frac{d^2 T}{dt^2} + \omega^2 T = 0; \quad \frac{r^2}{u} \left(\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \omega^2 u \right) = -\frac{1}{v} \frac{d^2 v}{d\theta^2}. \quad (7)$$

A solution of the first of these is $T = \cos(\omega t + \varphi)$, φ an arbitrary constant, and the second equation implies that both members are equal to a constant, which from physical conditions, is positive, say p^2 . Then one has $v = \cos(p\theta + \psi)$ and

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \left(\frac{\omega^2}{a^2} - \frac{p^2}{r^2} \right) u = 0 \quad (8)$$

which is essentially of the form (1.1), with the solution

$$u = AJ_p \left(\frac{\omega}{a} r \right) + BY_p \left(\frac{\omega}{a} r \right) \quad (9)$$

and the boundary conditions, $u(0)$ finite, $u(\rho) = 0$, where ρ denotes the radius of the membrane. Thus from these conditions one finally gets

$$z = A \cos(\omega t + \varphi) \cdot \cos(p\theta + \psi) \cdot J_p \left(\frac{\omega}{a} r \right), \quad J_p(\omega\rho/a) = 0. \quad (10)$$

EULER develops the power series for $J_p(\omega\rho/a)$ in ¶ 17 and states that the second equation in (10) has an infinite number of roots corresponding to each of the values $p = 0, 1, 2, \dots$. (The values of (r, θ) and $(r, \theta + 2n\pi)$ are equal so that p must be an integer for z to be single-valued and periodic.)

7. A problem in structural mechanics of Leonhard Euler

Early in his career, EULER had considered problems involving the bending of rods subjected to various types of loads which had already been investigated by JAMES BERNOULLI. In 1743, motivated by some questions of DANIEL BERNOULLI, he published a treatise concerned with the deflections of elastic bands which are loaded, and with the vibrations of such bands. But this work received comparatively little attention. In 1757 and 1778, EULER [2], [5] and [6] took up the problem of the buckling of a column under its own weight. The results which he obtained are fundamental in the theory of structural engineering.

If a short uniform rod or wire is clamped vertically, it will remain in a vertical position. But as the length is increased, a critical value is reached beyond which the rod tends to bend from the vertical. (A part of the weight of the rod acts through a moment arm which is proportional to the length by which the center of gravity has deviated from the vertical. This torque causes the rod to bend.) EULER considered the determination of this critical value, and assumed the hypothesis of JAMES BERNOULLI that the curvature of the rod at any point is proportional to the bending moment at that point.

Let l and w be the length and weight per unit length of the rod respectively and let β denote its flexural rigidity. Let OX denote the vertical axis through the rod (downwards) where O is at the upper end of the rod (in the vertical position), and let OY be taken horizontally in a plane through the rod.

Consider a point P on the rod with abscissa x and horizontal displacement from OX and Q a point (ξ, η) on the rod above P . Assume that $y' \ll 1$ so that, to a first order approximation, the curvature is y'' . It is proportional to the moment about P of the part of the rod above P . Thus

$$\beta \frac{d^2 y}{dx^2} = \int_0^x w(\eta - y) d\xi. \quad (1)$$

(Cf. EULER [5; ¶ 32].) Differentiate (1) with respect to x to obtain

$$\beta \frac{d^2 p}{dx^2} = \int_0^x w \left(-\frac{dy}{dx} \right) d\xi = -wxp, \quad p = dy/dx.$$

(Cf. EULER [2: ¶ 38].) On comparing the differential equation in p with VON LOMMEL's generalized form of BESSEL's differential equation (3.6), one gets

$$p = \sqrt{x} \{ AJ_{\frac{1}{3}}(2\lambda x^{3/2}) + BJ_{-\frac{1}{3}}(2\lambda x^{3/2}) \}, \quad \lambda^2 = \frac{1}{9} \frac{\omega}{\beta}, \quad (2)$$

where A and B are arbitrary constants. When the Bessel functions are expanded in series

$$J_{\pm 1/3}(2\lambda x^{3/2}) = (\lambda x^{3/2})^{\pm 1/3} \cdot \sum_{r=0}^{\infty} \frac{(-1)^r (\lambda^2 x^3)^r / r!}{(r \pm 1/3)!} \quad (2')$$

The boundary conditions are $p = 0$ when $x = l$ and $p' = 0$ when $x = 0$ since the bending moment is zero. The second condition implies $A = 0$, and the first condition can only be satisfied for $B \neq 0$ when

$$J_{-\frac{1}{3}}(2\lambda l^{3/2}) = 0 \quad (3)$$

The smallest value of l satisfying this equation is the critical value for the rod. If the length of the rod exceeds this, there will be a bend at the upper end. (The smallest zero of $J_{-\frac{1}{3}}(x)$ is about 1.86635.)

(A fascinating article by G. GREENHILL [1] extends EULER's results on the stability of vertical rods to those of variable cross-section subject to the influence of gravity.)

In the course of his long and uniquely prolific career, EULER developed many results involving Bessel functions in connection with problems of mechanics and solutions of certain types of differential equations. But his results were scattered over many memoirs with little or no interconnection and were occasionally rediscovered by others. It remained for his successors in the nineteenth century to recognize the value of developing a systematic theory of the properties of BESSEL functions.

J. L. L. LAGRANGE, P. S. LAPLACE and M. A. PARSEVAL were among other eighteenth century mathematicians who encountered BESSEL functions in their researches. (See WATSON [1]. An important astronomical problem treated by LAGRANGE will be discussed in Section 9.)

8. Analytical treatments of heat problems by Fourier and Poisson

The classical treatise of FOURIER [1], described by CLERK MAXWELL as “a great mathematical poem”, contains methods and results which have had a lasting influence on the development of mathematics and mathematical physics.

In Chapter VI he investigated an important problem in the diffusion of heat and extended the results for $J_0(x)$ developed by others. Consider a very long homogeneous right circular cylinder of radius a which has been gradually heated so that the temperature at all the points equidistant from the cylindrical axis is the same. The cylinder is put in a medium which is kept at zero temperature and the problem is to determine the distribution of temperature in the cylinder at a time t afterward.

Let v denote the temperature at time t of a point at a distance x from the axis. Consider a cross section of unit length and the ring bounded by the radii x and $x + dx$. In the time interval $(t, t + dt)$ the excess of the amount of heat flowing into the volume element over that flowing out is

$$2\pi K \left[x \frac{\partial v}{\partial x} + \frac{\partial}{\partial x} \left(x \frac{\partial v}{\partial x} dx - x \frac{\partial v}{\partial x} \right) \right] dt = 2\pi K \left(x \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x} \right) dx dt,$$

where K denotes the thermal conductivity of the cylinder. But this quantity is equal to the temperature increase times the volume element,

$$CD \cdot 2\pi x dx \cdot \frac{\partial v}{\partial t} dt,$$

where C is the specific heat and D the density. Thus one gets

$$\frac{\partial v}{\partial t} = \frac{K}{CD} \left(\frac{\partial^2 v}{\partial x^2} + \frac{1}{x} \frac{\partial v}{\partial x} \right); \quad 0 \leq x < a, \quad t > 0. \quad (1)$$

The quantity of heat flowing into the volume element in the interval dt is $2\pi x K (\partial v / \partial x) dt$, and at the surface $x = a$, the loss to the medium is $2\pi a \cdot h v dt$ where h is the external conductivity. (This is basically an extension of a law of

cooling due to NEWTON.) Thus one has the boundary condition

$$-K \frac{\partial v}{\partial x} = hv, \quad x = a \quad (2)$$

(FOURIER [1; Art. 118–120]). He substitutes $k = K/CD$ and assumes a particular solution of (1) in the form $v = u \exp(-mt)$ where m is a constant and u depends only on x . (Since the cylinder cools with time, $m > 0$.) On substituting the particular solution in (1), FOURIER [1; Art. 306] gets

$$\frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx} + \frac{m}{k} u = 0 \quad (3)$$

and the power series solution

$$u = 1 - \frac{gx^2}{2^2} + \frac{g^2 x^4}{2^2 \cdot 4^2} - \frac{g^3 x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots, \quad (4)$$

where $g = m/k$. Thus $u = J_0(x\sqrt{g})$. (The solution $Y_0(x\sqrt{g})$ may be disregarded since the temperature is finite when $x = 0$.) From the boundary condition (2), he gets the equivalent of

$$hJ_0(a\sqrt{g}) = -\sqrt{g}J_0'(a\sqrt{g}) \quad (5)$$

where h has been replaced by h/K , and proceeds to solve this equation in which both sides are power series in g . By applying ROLLE's theorem, he proves (in incomplete form) that there are an infinite number of roots and that they are all real. (Cf. BERNOULLI's assumption that $J_0(2c\sqrt{l}) = 0$ has an infinite number of real roots in Section 5.) Let g_1, g_2, g_3, \dots denote the roots and let u_1, u_2, u_3, \dots denote the values of u corresponding to these roots from (4). Then $u_m \exp(-g_m kt)$ is a solution, $m = 1, 2, 3, \dots$. The complete solution is of the form

$$v = \sum_{m=1}^{\infty} \alpha_m u_m \exp(-g_m kt), \quad (6)$$

and he shows in Art. 314–319 how they may be obtained.

FOURIER derives the equivalent of an integral representation previously obtained by M. A. PARSEVAL:

$$J_0(\alpha) = \frac{1}{\pi} \int_0^{\pi} \cos(\alpha \sin x) dx. \quad (7)$$

(See WATSON [1; 9].) This representation can be formally verified by setting $y = (1/\pi) \int_0^{\pi} \cos(x \sin t) dt$, computing y' , y'' and integrating by parts to show that $y + y'' = -(1/x)y'$, the equation satisfied by $J_0(x)$ in (1.1). He applies this result to the summation of the series for $u(a)$ obtained from (4) after substituting $ga^2/2^2 = \theta$. Thus

$$y = 1 - \theta + \frac{\theta^2}{(2!)^2} - \frac{\theta^3}{(3!)^2} + \dots = \frac{1}{\pi} \int_0^{\pi} \cos(\theta \sin x) dx. \quad (8)$$

Now $y = J_0(2\theta)$ satisfies the differential equation

$$\theta \frac{d^2 y}{d\theta^2} + \frac{dy}{d\theta} + y = 0. \quad (8')$$

Let $y^{(m)} = d^m y / d\theta^m$, $m = 0, 1, 2, \dots$. It follows from (8') that

$$-\frac{y^{(m)}}{y^{(m-1)}} = \frac{1}{1 + \theta y^{(m+1)}/y^{(m)}}, \quad m = 1, 2, 3, \dots$$

and thus he gets the continued fraction expansion

$$\frac{y'}{y} = \frac{-1}{1 - \frac{\theta}{2 - \frac{\theta}{3 - \frac{\theta}{4 - \dots}}}} \quad (9)$$

In 1811 FOURIER received a prize for his work, reviewed by LAGRANGE, LAPLACE and LEGENDRE, from the Académie des Sciences, although it was criticized for its lack of rigor and was not published until more than a decade later when FOURIER, then the Secretary of the Académie, incorporated it in his treatise (FOURIER [1]).

I. GRATTAN-GUINNESS [2; 376] in his biography of JOSEPH FOURIER claims that the name "Bessel function" is a misnomer and that it was FOURIER who first recognized its potentialities. Now virtually all of FOURIER's results, which deal only with properties of $J_0(x)$, were obtained by D. BERNOULLI, L. EULER, J. L. L. LAGRANGE, M. A. PARSEVAL and other eighteenth century predecessors. FOURIER's work lay *unpublished* in the archives of the Académie des Sciences in Paris for years after BESSEL [2] published his initial work on the functions named for him and FOURIER's results were unknown to BESSEL.

Even in the field of the development of special functions in trigonometric series, where FOURIER's results were most acclaimed, L. EULER, D. BERNOULLI and others obtained results previously on the trigonometric expansion of algebraic polynomials, as acknowledged by FOURIER himself, *etc.* The looseness of FOURIER's derivations, which led to their rejection by the reviewers of his memoir of 1811, has been criticized many times. For a recent example of these criticisms, see P. J. DAVIS & R. HERSH [1; 262 ff.].

S. D. POISSON (1781–1840) was FOURIER's great rival in mathematical physics in the early nineteenth century. He studied FOURIER's essay in the archives of the Académie. In 1823, POISSON [1] published a lengthy memoir which had initially been presented in 1815. The memoir consists of two parts, and in the second of these he develops some interesting representations of Bessel functions.

These results stem from an analysis of the flow of heat in a solid sphere and in a cylinder. For a homogeneous solid in which the temperature u at a point (x, y, z) is a function of the spatial coordinates and the time t , FOURIER [1; Art. 142–145] had considered a small rectangular parallelepiped centered at the point, and the heat flow through the three pairs of parallel faces. This is equal to the increase in temperature in the parallelepiped in the time dt . Thus he

obtained the heat equation

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (10)$$

where the positive constant $K = k/cd$, where k is the conductivity, c the specific heat, and d the density of the body. POISSON [1; 286 ff] considers the case of a homogeneous sphere which has been heated and is permitted to cool. He substitutes spherical coordinates in (10), multiplies u by r and obtains the transformed equation

$$\frac{\partial ru}{\partial t} = K \left[\frac{\partial^2 ru}{\partial r^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial ru}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 ru}{\partial \psi^2} \right] \quad (11)$$

with a boundary condition at $r = l$, the radius of the sphere,

$$\frac{\partial u}{\partial r} + b(u - \zeta) = 0, \quad (11')$$

where b is a positive constant which depends on the material of the sphere (and its surface) and ζ is the temperature of the medium in which the sphere cools. POISSON develops a lengthy solution of which only the final result can be given here. He expands u in a series of spherical harmonics in (θ, ϕ) , $4\pi ru = \sum_{n=0}^{\infty} (2n+1)V_n$ and, on p. 300, shows that $V_n = NR_n \exp - (K^2 \rho^2 t)$, where R_n satisfies the differential equation

$$\frac{d^2 R_n}{dr^2} + \rho^2 R_n - \frac{n(n+1)}{r^2} R_n = 0 \quad (12)$$

with a solution

$$R_n = r^{n+1} \int_0^{\pi} \cos(r\rho \cos \omega) \sin^{2n+1} \omega \, d\omega; \quad n = 0, 1, 2, \dots$$

(where ρ is a constant). This, up to a factor, is essentially equivalent to a particular case of LOMMEL's integral representation

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1/2)\sqrt{\pi}} \int_0^{\pi} \cos(z \cos \theta) \sin^{2\nu} \theta \, d\theta, \quad R(\nu) > -\frac{1}{2} \quad (13)$$

so that POISSON has obtained a representation for a Bessel function whose order is half an odd integer. (Cf. EULER [3; ¶ 1045].)

Similarly POISSON [1; 335ff] treats the problem of the distribution of heat in a finite homogeneous right circular cylinder of radius λ which has been heated and then permitted to cool. In (10), he substitutes cylindrical coordinates (s, ψ, x) and gets

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial s^2} + \frac{1}{s} \frac{\partial u}{\partial s} + \frac{1}{s^2} \frac{\partial^2 u}{\partial \psi^2} \right). \quad (14)$$

The cylinder is immersed in a medium of constant temperature which may be taken as zero. At the lateral surface of the cylinder where $s = \lambda$, one has the boundary condition

$$\frac{\partial u}{\partial s} + bu = 0 \tag{14'}$$

and at the bases where $x = l$ and $x = -l$, one has

$$\frac{\partial u}{\partial x} + \beta u = 0, \quad \frac{\partial u}{\partial x} + \beta' u = 0 \tag{14''}$$

respectively. In solving this, POISSON [1; 340] obtains the integral

$$R_n = s^{n+\frac{1}{2}} \int_0^\pi \cos(s \cdot h \cos \omega) \cdot \sin^{2n} \omega \, d\omega \tag{15}$$

where h is a variable which assumes positive values.

POISSON investigates the particular cases in which the radius of the cylinder is very small and very large. For the latter case, POISSON [1; 346 ff] is led to consider the approximations

$$y = \frac{1}{\pi} \int_0^\pi \cos(k \cos \omega) \, d\omega = \sum_{m=0}^\infty \frac{(-1)^m (k/2)^m}{(m!)^2}.$$

(Cf. (7) above. It may be proved in an analogous manner.) POISSON shows that y satisfies the differential equation for $J_0(k)$. The smaller roots of $y = 0$ can be obtained from the series expansion but this is impractical for large k . He rewrites the differential equation for y in the form

$$\frac{d^2 y \sqrt{k}}{dk^2} + \left(\frac{1}{4k^2} + 1 \right) y \sqrt{k} = 0. \tag{16}$$

For large values of k , the term involving $1/4k^2$ can be neglected and the equation can be approximated by that for a harmonic oscillator. POISSON gives a solution for (16), i.e. $J_0(k)$ in the form

$$y \sqrt{k} = \left(\sum_{m=0}^\infty \frac{A_m}{k^m} \right) \cos k + \left(\sum_{m=0}^\infty \frac{B_m}{k^m} \right) \sin k \tag{17}$$

and proceeds to determine the values of A_m and B_m in terms of A_0 and B_0 . From the recursion relations for the coefficients in (17) we have:

$$\begin{aligned} 2(m+1)A_{m+1} + [1/4 + m(m+1)]B_m &= 0; \\ -2(m+1)B_{m+1} + [1/4 + m(m+1)]A_m &= 0 \end{aligned}$$

for $m = 0, 1, 2, 3, \dots$. The constants A_0 and B_0 can be determined from the values of $y \sqrt{k}$ and $dy \sqrt{k}/dk$ as $k \rightarrow \infty$. Thus

$$A_0 = \left(y \cos k - \frac{dy}{dk} \sin k \right) \sqrt{k}; \quad B_0 = \left(y \sin k + \frac{dy}{dk} \cos k \right) \sqrt{k}.$$

POISSON substitutes the integral for y (after (15)) here and obtains

$$A_0 = \sqrt{k} \int_0^\pi \left[\cos\left(2k \sin^2 \frac{\omega}{2}\right) \cdot \cos^2 \frac{\omega}{2} + \sin\left(2k \cos^2 \frac{\omega}{2} \cdot \sin^2 \frac{\omega}{2}\right) \right] d\omega$$

$$B_0 = \sqrt{k} \int_0^\pi \left[\sin\left(2k \sin^2 \frac{\omega}{2}\right) \cdot \cos^2 \frac{\omega}{2} + \sin\left(2k \cos^2 \frac{\omega}{2} \cdot \sin^2 \frac{\omega}{2}\right) \right] d\omega.$$
(18)

After lengthy reductions, he lets $k \rightarrow \infty$ and finds $A_0 = B_0 = \sqrt{\pi}$.

A concise demonstration of POISSON's results is given by WATSON [1; 11–12].

9. On an astronomical problem of Kepler and some trigonometric series solutions

In 1609, J. KEPLER (1571–1630) published his famous treatise [1], in which he gave empirical laws summarizing the results of many years of planetary observations. In Newtonian terms, there were two laws – the first was to the effect that the orbits of planets are ellipses with the sun at one focus. The second law states that the radius vector, connecting the sun and a planet, sweeps out equal areas in equal times. About sixty years later, I. NEWTON derived KEPLER's empirical laws as a consequence of the law of inverse squares in his theory of gravitation.

KEPLER developed a remarkable equation in Chapter LX of his treatise which subsequently generated a large literature. Before taking up the specific contributions of J. L. L. LAGRANGE and F. W. BESSEL to the solution of KEPLER's equation, it is necessary to develop some of the astronomical background.

Consider a planet moving in an ellipse with parametric coordinates $(a \cos E, b \sin E)$ about a center $C:(0,0)$. The sun S is at a focus $(ae,0)$ where e is the eccentricity of the ellipse, and $A:(a,0)$ and $A':(-a,0)$ denote the perihelion and aphelion respectively. Let P denote the position of the planet in its orbit at a particular time t measured from the instant when the planet passes through A . With the ellipse there is associated a circumscribed circle with parametric coordinates $(a \cos E, a \sin E)$, tangent to the ellipse at A and A' , and with P there is associated a corresponding point Q on the circle at time t . The angles $ACQ = E$ and $ASP = W$ are called the *eccentric anomaly* and *true anomaly*, respectively. Finally, $M = 2\pi t/T$, where T is the period of revolution of the planet, is called the *mean anomaly*, so that M can be expressed in terms of time t .

Now

$$\frac{t}{T} = \frac{\text{area of elliptical sector } ASP}{\text{area of ellipse}} = \frac{\text{area of } ASP}{\pi ab}.$$

By orthogonal projection,

$$\frac{t}{T} = \frac{\text{area of circular sector } ASQ}{\text{area of circle}} = \frac{\text{area of } ASQ}{\pi a^2}.$$

and

$$\text{area of } ASP = b(\text{area of } ACQ - \text{area of } \Delta SCQ)/a$$

Thus

$$\frac{t}{T} = \frac{M}{2\pi} = \frac{(a^2 E - a^2 e \sin E)}{2\pi a^2} = \frac{E - e \sin E}{2\pi}$$

or

$$M = E - e \sin E \tag{1}$$

This is KEPLER's equation. The problem is to solve for E , the eccentric anomaly, in terms of M , the mean anomaly. (Moreover, the radius vector SP and the true anomaly W can also be expressed in terms of E , so that all the geometric elements of the orbit can be expressed as functions of E and thence as functions of M once (1) is solved.)

KEPLER believed that the equation could not be solved exactly because of the heterogeneity of E and $\sin E$. His method of obtaining an approximate solution was the onerous one of calculating tables of the mean anomaly M and also of the true anomaly W for integral values of the eccentric anomaly E , and using interpolation to obtain intermediate values. Numerous successors in the seventeenth and eighteenth centuries gave approximate solutions of greater or lesser value. (E.g. ISAAC NEWTON in his treatise *Principia Mathematica* of 1687 gave a graphical construction for a cycloid (Book 1, Prop. XXXI) stemming from CHRISTOPHER WREN from which an approximate solution for E in terms of M can be readily obtained.)

In 1770, J. L. L. LAGRANGE (1736–1813) gave an analytical solution of the KEPLER equation [1]. He developed a theorem for the expansion of a function, defined by an implicit equation, in a series. Let $y = x + a\varphi(y)$, where a is a parameter and $\varphi(y)$ has a power series expansion. Let $f(y)$ be a function which has a power series expansion. Then for a sufficiently small, $f(y)$ may be expanded in a convergent series

$$f(y) = f(a) + \sum_{n=1}^{\infty} \frac{a^n}{n!} \frac{d^{n-1}}{da^{n-1}} [f'(a) \cdot (\varphi(a))^n]. \tag{2}$$

(See e.g. WHITTAKER & WATSON [1; 132–133].) On applying his theorem to (1) he found the formal solution

$$E = M + \sum_{n=1}^{\infty} \frac{e^n}{n!} \frac{d^{n-1}}{dM^{n-1}} [\sin^n M]. \tag{3}$$

The first few terms, which usually suffice for applications, are

$$E = M + (e - \frac{1}{8}e^3) \sin M + \frac{1}{2}e^2 \sin 2M + \frac{3}{8}e^3 \sin 3M + \dots \tag{3'}$$

[Let r denote the radius vector SP . Then $r^2 = (ae - a \cos E)^2 + b^2 \sin^2 E$ where $r = a(1 - e \cos E)$. From this and (2) it follows that r/a can be determined in terms of M by setting $f(E) = 1 - e \cos E$. Similarly, the true anomaly, W , can be obtained from $\tan W = b \sin E / (a \cos E - ae)$. For on getting the equation $(1 - \cos W) / (1 + \cos W) = (1 - e)(1 - \cos E) / (1 + e)(1 + \cos E)$ from this, and noting that $\tan^2(W/2) = (1 - \cos W) / (1 + \cos W)$ etc. one finds $\tan^2(W/2) = (1 - e) \tan^2(E/2) / (1 + e)$. Thus $f(E) = \sqrt{(1 - e) / (1 + e)} \cdot \tan E/2$, in (2).]

But in (3), the computation of $\sin^n M$ in terms of sines or cosines of multiples of M to obtain the derivatives, is laborious and LAGRANGE was led to consider the series for E and r in the forms

$$E - M = \sum_{n=1}^{\infty} A_n \sin nM, \quad \frac{r}{a} = 1 + e^2/2 + \sum_{n=1}^{\infty} B_n \cos nM \tag{4}$$

where

$$\begin{aligned} \frac{A_n}{2} &= \left(\frac{e}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m n^{2m+n-1}}{m!(n+m)!} \left(\frac{e}{2}\right)^{2m}, \\ -\frac{B_n}{2} &= \left(\frac{e}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)^{2m+n-2}}{m!(n+m)!} \left(\frac{e}{2}\right)^{2m}. \end{aligned} \tag{5}$$

LAGRANGE obtained the equivalents of A_n and B_n for $n = 0, 1, 2, 3$. From (2.10) one sees that

$$A_n = 2J_n(ne)/n, \quad B_n = -2(e/n)J'(ne). \tag{5'}$$

(See WATSON [1; 6].)

The method employed by LAGRANGE for obtaining (5) from (4) is indirect and complicated, particularly for n even moderately large. In 1817, F. CARLINI (1783–1862) gave a remarkable asymptotic approximation for A_n , [1], for the case where n is large. Also, among the papers left by P. S. LAPLACE (1749–1827) and incorporated in his great treatise [1; Tome 5, Supplement, Sect. 2] there is an asymptotic formula for B_n . (Cf. WATSON [1; 6–9].)

BESSEL [2] states that he had seen the solution to KEPLER's problem given in LAGRANGE's *Mécanique Analytique*, but he had found it clumsy. As a result, he developed a different method of solution involving infinite trigonometric series whose coefficients yield the functions since named for him.

Before treating BESSEL's solution, it is useful to consider his prior application of finite trigonometric series to the reduction of data obtained from astronomical instruments.

One has for integers $p, n, n \neq 0$:

$$\sum_{v=1}^{n-1} \sin \frac{2pv\pi}{n} = 0, \quad \sum_{v=1}^{n-1} \cos \frac{2pv\pi}{n} = \begin{cases} 0 & \text{for } p \not\equiv 0 \pmod{n}, \\ n & \text{for } p \equiv 0 \pmod{n}. \end{cases} \quad (6)$$

These results have been known for centuries and can readily be proved analytically. (From a geometric standpoint n uniformly distributed particles on a unit circle can be regarded as the vertices of a regular polygon whose center of gravity is at the center of the circle for $n > 1$.) By simple trigonometric reductions it follows that if q is an integer, one gets the orthogonality relations

$$\begin{aligned} \sum_{v=1}^{n-1} \sin \frac{2pv\pi}{n} \cos \frac{2qv\pi}{n} &= 0, \\ \sum_{v=1}^{n-1} \cos \frac{2pv\pi}{n} \cos \frac{2qv\pi}{n} &= \sum_{v=1}^{n-1} \sin \frac{2pv\pi}{n} \sin \frac{2qv\pi}{n}, \\ &= \begin{cases} 0, & \text{if neither } p+q \text{ nor } p-q \equiv 0 \pmod{n} \\ \frac{n}{2} \text{ or } \pm \frac{n}{2} & \text{if } p-q \equiv 0 \text{ or } p+q \equiv 0 \pmod{n}. \\ n \text{ or } 0, & \text{if } p+q \text{ and } p-q \equiv 0 \pmod{n} \end{cases} \end{aligned} \quad (6')$$

BESSEL's approach to the determination of the coefficients of a finite trigonometric series to represent a function depended on the application of the method of least squares. As early as 1813 in a letter to GAUSS, he outlined this method and in BESSEL [1] and earlier astronomical journals, he applied the method to the graduation of a circle in an astronomical instrument. Let the representation of $y(t)$ be

$$y(t) = \frac{\alpha_0}{2} + \sum_{\mu=1}^n [\alpha_\mu \cos \mu t + \beta_\mu \sin \mu t], \quad (7)$$

and solve for the coefficients in the series

$$y_v = \frac{a_0}{2} + \sum_{\mu=1}^m \left[a_\mu \cos \frac{2\mu v\pi}{n} + b_\mu \sin \frac{2\mu v\pi}{n} \right]; \quad \mu = 0, 1, 2, \dots, n-1, \quad (8)$$

where y_v denotes the observational data and the number of equations n exceeds the number $2m + 1$ of unknown coefficients. The "best" values in the sense of the method of least squares depend on minimizing the sum of squares,

$$\sum_{v=0}^{n-1} \left[y_v - \frac{a_0}{2} - \sum_{\mu=1}^m \left(a_\mu \cos \frac{2\mu v\pi}{n} + b_\mu \sin \frac{2\mu v\pi}{n} \right) \right]^2 \equiv \sum_{v=0}^{n-1} \left[y_v - S_m \left(\frac{2v\pi}{n} \right) \right]^2 \geq 0,$$

with respect to the coefficients. On differentiating one finds

$$\begin{aligned} \frac{\partial}{\partial a_\mu} \sum_{v=0}^{n-1} \left[y_v - S_m \left(\frac{2v\pi}{n} \right) \right]^2 &= -2 \sum_{v=0}^{n-1} y_v \cdot \cos \frac{2\mu v\pi}{n} + 2 \sum_{v=0}^{n-1} S_m \left(\frac{2v\pi}{n} \right) \cdot \cos \frac{2\mu v\pi}{n}, \\ \frac{\partial}{\partial b_\mu} \sum_{v=0}^{n-1} \left[y_v - S_m \left(\frac{2v\pi}{n} \right) \right]^2 &= -2 \sum_{v=0}^{n-1} y_v \cdot \sin \frac{2\mu v\pi}{n} + 2 \sum_{v=0}^{n-1} S_m \left(\frac{2v\pi}{n} \right) \cdot \sin \frac{2\mu v\pi}{n}. \end{aligned}$$

On setting each equation equal to zero and using the orthogonality relations (6'), one gets

$$a_\mu = \frac{2}{n} \sum_{\nu=0}^{n-1} y_\nu \cos \frac{2\mu\nu\pi}{n}, \quad b_\mu = \frac{2}{n} \sum_{\nu=0}^{n-1} y_\nu \sin \frac{2\mu\nu\pi}{n}; \quad 0 \leq \mu \leq m \quad (9)$$

so that all the coefficients in $S_m(\frac{2\nu\pi}{n})$ are determined. The second derivatives of the sum $\sum_{\nu=0}^{n-1} [y_\nu - S_m(\frac{2\nu\pi}{n})]^2$ with respect to the individual coefficients are all positive so that the values given by (9) yield a minimum. It follows that the minimum value of the sum $\sum_{\nu=0}^{n-1} [y_\nu - S_m(\frac{2\nu\pi}{n})]^2 = \sum_{\nu=0}^{n-1} y_\nu^2 - \frac{n}{2} \left[\frac{a_0^2}{2} + \sum_{\mu=1}^m (a_\mu^2 + b_\mu^2) \right]$.

BESSEL found the important property of "finality" of the coefficients determined by the application of the method of least squares to trigonometric polynomials and which is a consequence of the orthogonality relations (6'). The values of the coefficients in (9) remain *unchanged* when m is replaced by $m + 1$. Only the new coefficients a_{m+1} , b_{m+1} have to be calculated. (See BURKHARDT & ESCLANGON [1] for further details.)

The method of developing formulas for the case of an "arbitrary" periodic continuous function $f(t)$, of period 2π , approximated by a finite trigonometric polynomial $S_m(t) = (a_0/2) + \sum_{\mu=1}^m [a_\mu \cos \mu t + b_\mu \sin \mu t]$ proceeds in an analogous manner. If p and q are non-negative integers, one gets the orthogonality conditions

$$\int_0^{2\pi} \sin pt \cdot \cos qt \, dt = 0, \quad \int_0^{2\pi} dt = 2\pi, \quad (10)$$

$$\int_0^{2\pi} \cos pt \cdot \cos qt \, dt = \int_0^{2\pi} \sin pt \cdot \sin qt \, dt = \begin{cases} 0 & \text{for } p \neq q \\ \pi & \text{for } p = q. \end{cases}$$

By minimizing $\int_0^{2\pi} [f(t) - S_m(t)]^2 \, dt$, one gets

$$a_\mu = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos \mu t \, dt, \quad b_\mu = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin \mu t \, dt. \quad (11)$$

The minimum value of the error is non-negative and from (10) and (11)

$$\int_0^{2\pi} [f(t) - S_m(t)]^2 \, dt = \int_0^{2\pi} [f(t)]^2 \, dt - \pi \left[\frac{a_0^2}{2} + \sum_{\mu=1}^m [a_\mu^2 + b_\mu^2] \right]$$

and thus one gets BESSEL's inequality

$$\int_0^{2\pi} [f(t)]^2 \, dt \geq \pi \left[\frac{a_0^2}{2} + \sum_{\mu=1}^m [a_\mu^2 + b_\mu^2] \right]. \quad (12)$$

BESSEL's condition of finality holds. If one lets $m \rightarrow \infty$, from the Dirichlet conditions one gets

$$f(t) = \frac{a_0}{2} + \sum_{\mu=1}^{\infty} [a_\mu \cos \mu t + b_\mu \sin \mu t]. \quad (13)$$

(The DIRICHLET conditions, which are sufficient for the validity of the representation (13), were discovered in 1829, after the publication of the work of

BESSEL and FOURIER, and can be found in textbooks on FOURIER series. Actually, the condition that $f(t)$ be continuous in $[0, 2\pi]$ can be relaxed considerably.)

BESSEL's solution of the KEPLER problem [2] will now be discussed. He states that the (arbitrary periodic) function U (of period 2π) can be represented by (the infinite trigonometric series)

$$U = \sum_{n=1}^{\infty} (A_n \sin nu + B_n \cos nu) \tag{14}$$

where

$$A_n = \frac{1}{\pi} \int_0^{2\pi} U \cdot \sin nu \, du, \quad B_n = \frac{1}{\pi} \int_0^{2\pi} U \cdot \cos nu \, du. \tag{15}$$

(Cf. (14) and (11).) He does not discuss the conditions under which (14) and (15) hold, but says that (15) can be obtained either by finite integrations or by application of a method which he had previously used in reducing observational data (the method of least squares).

BESSEL's method of developing his results is lengthy, and a simpler derivation will be given here. From (1), $E - M$ is a periodic function which vanishes when M is a multiple of π (at perihelion and aphelion). From (14) and (15)

$$E - M = \sum_{n=1}^{\infty} A_n \sin nM$$

where

$$\frac{\pi}{2} A_n = \int_0^{\pi} (E - M) \sin nM \, dM.$$

On integrating by parts, one has

$$\frac{\pi}{2} A_n = \int_0^{\pi} \frac{\cos nM}{n} \, dE = \frac{1}{n} \int_0^{\pi} \cos n(E - e \sin E) \, dE. \tag{16}$$

It may be shown that $\pi n A_n / 2$, a function of n and e , satisfies the differential equation (1.1) and thus that $A_n = 2J_n(ne)/n$. This agrees with (5').

Similarly, from the discussion following (3') above, it is seen that radius vector r is an *even* periodic function of M , with period 2π , and by an integration by parts, one has

$$\frac{\pi}{2} B_n = \int_0^{\pi} (1 - e \cos E) \cdot \cos nM \, dM = \frac{-e}{n} \int_0^{\pi} \sin E \cdot \sin n(E - e \sin E) \, dM. \tag{17}$$

From this, as above, it follows that $B_n = -2eJ'(ne)/n$ as in (5').

BESSEL also gave an expansion for the difference between the true anomaly and the mean anomaly, *i.e.*, the "equation of the center",

$$W - M = \sum_{n=1}^{\infty} C_n \sin nM$$

where, on integration by parts,

$$\frac{\pi}{2} C_n = \int_0^\pi (W - M) \sin nM \, dM = \frac{\sqrt{1-e^2}}{n} \int_0^\pi \frac{\cos n(E - e \sin E)}{1 - e \cos E} \, dE. \quad (18)$$

BESSEL evaluated this integral as a power series in e by a complicated method. This was considerably simplified in BESSEL [3] by expressing $\sqrt{1-e^2}/(1-e\cos E)$ as a power series in $f = e/(1 + \sqrt{1-e^2})$ of the form

$$\frac{\sqrt{1-e^2}}{1-e\cos E} = 1 + 2 \sum_{m=1}^{\infty} f^m \cos mE$$

The coefficient of f^m reduces to $\pi[J_{n-m}(ne) + J_{n+m}(ne)]/n$.

It is worthwhile to compare the work of FOURIER and BESSEL in the development of periodic functions in infinite trigonometric series. The development of a function in an infinite series of multiples of cosines was considered by CLAIRAUT in 1756, and a corresponding series for multiples of sines by LAGRANGE between 1762 and 1765 (WHITTAKER & WATSON [1; 166]). A development of the cosine series in a form similar to that later given by FOURIER was developed by EULER in 1777 and published posthumously in 1793. In the first decade of the nineteenth century FOURIER and C. F. GAUSS (1777–1855) came upon “Fourier series” at about the same time but their work remained unpublished. (See BURKHARDT [2; 892, 928].)

The work of FOURIER remained unpublished until 1822. (FOURIER [1] and I. GRATAN-GUINNESS [2].) But in 1816, BESSEL developed “Fourier series” *independently* and by a *different* method and this work was published in 1819 and earlier in astronomical journals. *Now it is a well accepted principle in scientific research that publication establishes priority. By this criterion BESSEL’s achievement deserves adequate recognition.*

For further details on the justification of BESSEL’s method of developing infinite trigonometric series see A. SOMMERFELD [1, Ch. 1], M. BÖCHER [1] and H. DAVIS [1].

10. Bessel’s systematic investigation of his functions

In considering the perturbations of the elliptic orbit of a planet in the solar system by another planet, it is convenient to consider the direct effect of the perturbing planet and the indirect effect due to the motion of the sun caused by the perturbing planet. These effects were investigated by A. C. CLAIRAUT, who was successful in predicting the return of HALLEY’s comet to perihelion, with an error of about a month, after some extremely laborious calculations, and by J. D’ALEMBERT and L. EULER, among others in the mid-eighteenth century. BESSEL [3] set himself the task of investigating the indirect effect of the sun’s motion, and after a detailed discussion of the astronomical considerations involved was led to a systematic treatment of the functions which had already

arisen in his analysis of the KEPLER equation. (See BESSEL [2].)

BESSEL investigated properties of the functions

$$J_h(k) = \frac{1}{2\pi} \int_0^{2\pi} \cos(hE - k \sin E) dE; \quad h \text{ an integer} \quad (1)$$

which he had earlier encountered. (See (9.16).) Actually BESSEL used the notation I_k^h for the modern notation $J_h(k)$. If ones sets $y = \int_0^\pi \cos n(\varphi - x \sin \varphi) d\varphi$, then after an integration by parts of dy/dx with respect to φ , one gets

$$\begin{aligned} \frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} &= -n^2 y + \frac{n^2}{x} \int_0^\pi \cos \varphi \cdot \cos n(\varphi - x \sin \varphi) d\varphi \\ &= -n^2 y + \frac{n}{x^2} [\sin n(\varphi - x \sin \varphi)]_0^\pi + \frac{n^2}{x^2} y \end{aligned}$$

or

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + n^2 \left(1 - \frac{1}{x^2}\right) y = 0,$$

which, on replacing nx by x yields BESSEL's differential equation (1.1). BESSEL gave the series development

$$J_h(k) = \left(\frac{k}{2}\right)^h \sum_{n=0}^\infty \frac{(-1)^n}{n!(k+n)!} \left(\frac{k}{2}\right)^{2n} \quad (1')$$

and the integral evaluations

$$\begin{aligned} \frac{e}{2\pi} \int_0^{2\pi} \cos hM \cos E dE &= J_h(he), \\ \frac{1}{2\pi} \int_0^{2\pi} \sin hM \sin E dE &= \frac{J_h(he)}{e} - J_{h+1}(he) \end{aligned} \quad (2)$$

as well as the recursion formulas developed from

$$\bar{M} = \frac{e}{2\pi} \int_0^{2\pi} \frac{\cos hM \cdot \cos E dE}{1 - e \cos E}, \quad \bar{M}' = \frac{e}{2\pi} \int_0^{2\pi} \frac{\sin hM \cdot \sin E dE}{1 - e \cos E},$$

i.e.,

$$\frac{2\bar{M}}{e} = J_{h-1}(he) + J_{h+1}(he), \quad \frac{2\bar{M}'}{e} = J_{h-1}(he) + J_{h+1}(he). \quad (3)$$

From (2), he obtained expansions for

$$\cos W, \sin W, r \cos W, r \sin W, (\cos W)/r^2, (\sin W)/r^2.$$

Similarly, from (2) and (3) there follow expansions for

$$\log r, r^n, r^n \cos mW, r^n \sin mW, r^n \cos mE, r^n \sin mE.$$

BESSEL also obtained the equivalents of the recurrence relations (2.4) from which the continued fraction

$$\frac{J_h(k)}{J_{h-1}(k)} = \frac{k/2h}{1 - kJ_{h+1}(k)/2hJ_h(k)} = \frac{k/2h}{1 - \frac{k^2/4h(h+1)}{1 - \frac{k^2/4(h+1)(h+2)}{1 - \dots}}} \quad (4)$$

follows, as well as the differential equation

$$\frac{d^2}{dk^2} J_h(k) + \frac{1}{k} \frac{dJ_h(k)}{dk} + J_h(k) \left(1 - \frac{h^2}{k^2}\right) = 0. \quad (5)$$

From the TAYLOR series expansion

$$J_h(k+z) = \left(1 + \frac{z}{k}\right)^h \left[J_h(k) - \frac{z}{1!} \left(1 + \frac{z}{2k}\right) J_{h+1}(k) + \frac{z^2}{2!} \left(1 + \frac{z}{2k}\right)^2 J_{h+2}(k) - \dots \right] \quad (6)$$

BESSEL also derived a second integral representation

$$J_h(k) = \frac{k^h}{(2h-1)!!} \frac{1}{2\pi} \int_0^{2\pi} (\cos E)^{2h} \cos(k \sin E) dE. \quad (7)$$

(Cf. POISSON's integral representation (8.15).)

BESSEL concluded his memoir with a table of $J_0(k)$ and $J_1(k)$ to ten decimals from $k=0$ to $k=3.2$ with an interval 0.01 using (7).

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